ON THE GLOBAL WELL-POSEDNESS OF THE ONE-DIMENSIONAL SCHRÖDINGER MAP FLOW

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ABSTRACT. We establish the global well-posedness of the initial value problem for the Schrödinger map flow for maps from the real line into Kähler manifolds and for maps from the circle into Riemann surfaces. This partially resolves a conjecture of W.-Y. Ding.

1. Introduction

In this article we study the Schrödinger map flow from a one-dimensional domain into a complete Kähler manifold. First, we show that when the domain is the real line the flow exists for all time. Second, we show that when the domain is the circle and the target is a Riemann surface the flow also exists for all time. The main contribution of this article is to bring Bourgain's work on the periodic cubic nonlinear Schrödinger equation (NLS) to bear on the geometric situation at hand.

Let (M,g) be a complete Riemannian manifold of dimension m, and let (N,ω,J,h) be a complete symplectic manifold of dimension 2n with a compatible almost complex structure J, that is such that $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$ and such that $h(\cdot,\cdot)=\omega(\cdot,J\cdot)$ defines a complete Riemannian metric on N. Associated to this data is the space of all smooth maps from M to N, the Fréchet manifold $X:=C^\infty(M,N)$, endowed with a symplectic structure,

$$\Omega(V,W)|_{u} = \int_{M} u^{\star} \omega(V,W) dV_{M,g}, \quad \forall \ V,W \in T_{u}X = \Gamma(M,u^{\star}TN),$$

where the tangent space to X at a map $u: M \to N$ is the space of smooth sections of $u^*TN \to M$ and where $dV_{M,g}$ denotes the volume form on M induced by g. The form Ω is non-degenerate, i.e., endows X with an injective map $TX \to T^*X$.

Define the energy function on X by

$$E(u) = \frac{1}{2} \int_{M} |du|_{g^{\sharp} \otimes u^{\star}h}^{2} dV_{M,g},$$

where we denote by g^{\sharp} the metric induced by g on $T^{\star}M$ and where we view du as a section of $T^{\star}M \otimes u^{\star}TN \to M$ and equip this bundle with the metric $g^{\sharp} \otimes u^{\star}h$.

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The almost-complex structure on N induces one on X and a corresponding compatible Riemannian metric

$$G(V,W)|_{u} = \int_{M} u^{\star}h(V,W)dV_{M,g}, \quad \forall V,W \in T_{u}X = \Gamma(M,u^{\star}TN).$$

In the infinite-dimensional setting not every function will necessarily have a gradient. However, if we let $\{u_t\}_{t\in[-1,1]}$ be a smooth family of maps with $u_0=u$ and denote by $W=\frac{\partial u_t}{\partial t}\big|_0$ a variation, then

$$dE(W)|_{u_0} = \frac{\partial E(u_t)}{\partial t}\Big|_0 = \int_M g^{\sharp} \otimes u^{\star} h(du, dW) dV_{M,g}$$

$$= -\int_M u^{\star} h(\operatorname{tr}_{g^{\sharp}} \nabla du, W) dV_{M,g},$$
(1)

and hence the gradient of E exists and is given by

$$\nabla^G E|_u = -\tau(u),\tag{2}$$

where $\tau(u) := \operatorname{tr}_{g^{\sharp}} \nabla du$ is called the tension field of u and ∇ is the connection on $T^{\star}M \otimes u^{\star}TN \to M$ induced from the Levi-Civita connection on (M,g) and the pulled-back one from (N,h). The corresponding gradient flow

$$\frac{\partial u}{\partial t} = \tau(u), \quad u|_{\{0\} \times M} = u_0, \tag{3}$$

is the classical harmonic map flow introduced by Eells and Sampson [ES] that has been extensively studied.

Now the symplectic gradient of E also exists and is given by

$$\nabla^{\Omega} E|_{u} = -J\tau(u).$$

The corresponding Hamiltonian flow on (X,Ω) , introduced by Ding-Wang and Terng-Uhlenbeck [DW1, TU],

$$\frac{\partial u}{\partial t} = -J\tau(u), \quad u|_{\{0\}\times M} = u_0, \tag{4}$$

is called the Schrödinger map flow.

While the energy decreases along (3), for (4) the flow is contained in an energy level-set since for maps of finite energy we have by (1)

$$\frac{dE(u(t))}{dt} = -\int_{M} u^{\star} h(\tau(u), \frac{\partial u}{\partial t}) dV_{M,g} = 0.$$
 (5)

For (3) one typically expects to converge to a harmonic representative of the homotopy class of u_0 under some geometric assumptions (e.g., negatively curved target [ES]) while (4) seems to be describing some rather very different behavior. Analytically this may be described by the transition from the parabolic equation (3) to the borderline case (4) whose symbol has purely imaginary eigenvalues. Note also that for the Schrödinger flow there is no preferred time direction.

One problem common to both flows is the question of existence and uniqueness. Indeed since the flows are defined on infinite-dimensional spaces one cannot expect global existence/well-posedness in general.¹ Restricting to the Kähler case, there is a similarity between the two as far as the local existence is concerned: Results of Ding-Wang and McGahagan show that at least locally (4) can be approximated by equations of either parabolic (in the sense of Petrovskii (see, e.g., [EZ])) or hyperbolic character. As a consequence the following result holds for maps of finite energy.

THEOREM 1.1. (See [DW2, M2].) Let (M^m, g) be a complete Riemannian manifold and let (N, J, h) be a complete Kähler manifold with bounded geometry. For integers k > m/2 + 1 equation (4) with $u_0 \in W^{k,2}(M, N)$ admits a unique solution $u \in C^0([0,T], W^{k,2}(M,N))$ where $T < T_0$ and T_0 depends on $||\nabla u_0||_{W^{[m/2]+1,2}}$ and the geometry of N alone. Moreover, there exist positive constants C_1, C_2 depending only on these quantities such that

$$||\nabla u(t)||_{W^{[m/2]+1,2}} \le C_1/(T_0-t)^{C_2}, \quad \forall t \in [0, T_0).$$

In particular, if $u_0 \in C^{\infty}(M, N)$ then $u \in C^{\infty}([0, T] \times M, N)$.

Here by bounded geometry we mean uniform bounds on the injectivity radius and the curvature tensor and its derivatives. This is automatically true for compact targets.

The main difficulty lies, therefore, in understanding the global behavior.

Previous results of a global nature are mostly concerned with the one-dimensional domain case and are all restricted to the case of a special target Kähler manifold. We recall the following non-exhaustive list of works. The flow on $(S^1, \operatorname{can}) \to (S^2, \operatorname{can})$ corresponding to the classical model for an isotropic ferromagnet was studied from the mathematical point of view by Sulem, Sulem and Bardos [SSB] who obtained local well-posedness for the initial value problem as well as partial global results. Zhou, Guo and Tan [ZGT] studied the global well-posedness problem using a parabolic approximation which was later put to use by Ding and Wang [DW1] as well as by Pang, Wang and Wang [PWW1] to prove global existence and uniqueness of smooth solutions of maps from (S^1, can) to any constant sectional curvature Kähler target as well as to Hermitian locally symmetric spaces [PWW2] using a conservation law. The latter also treats the inhomogeneous flow which can be essentially viewed as the Schrödinger flow with domain S^1 equipped with a different metric. Terng and Uhlenbeck [TU] gave a detailed study of the flow from the Euclidean line into Grassmannians. Chang, Shatah and Uhlenbeck [CSU] proved existence and uniqueness of global smooth solutions for maps of the Euclidean line into a compact Riemann surface. In addition, they treated maps of the Euclidean plane into a compact Riemann surface under the assumption of small initial energy and certain symmetries. Finally, see Bejenaru, Ionescu, Kenig and Tataru [BIK, BIKT] for recent work on global well-posedness in the case of maps from Euclidean space into (S^2, can) under a certain smallness assumption.

¹ We note that for the Schrödinger flow the question of global existence is equivalent to the existence of a "symmetry" of (X,Ω) , i.e. a one-parameter subgroup of Hamiltonian diffeomorphisms of (X,Ω) for the energy function E integrating $\nabla^{\Omega}E$.

² Until recently no results were known for general symplectic targets; see Chihara [Ch] for recent work on local well-posedness in this setting.

Note that in all of these results one restricts the target to a rather small class of Kähler manifolds. We recall the following conjecture of Ding.

Conjecture 1.2. (See [D].) The Schrödinger map flow is globally well-posed for maps from one-dimensional domains into compact Kähler manifolds.

The main results of this article are a partial answer to this conjecture. Namely, we establish the global well-posedness of the one-dimensional Schrödinger flow into general Kähler manifolds when the domain is the real line, and into Riemann surfaces when the domain is the circle.

THEOREM 1.3. Let $(M,g) = (\mathbb{R}, dx \otimes dx)$, let (N,J,h) be a complete Kähler manifold with bounded geometry, and let $k \geq 2$ be an integer. The flow equation (4) with $u_0 \in W^{k,2}(\mathbb{R},N)$ admits a unique solution $u \in C^0(\mathbb{R}, W^{k,2}(\mathbb{R},N))$. In particular u is smooth if u_0 is.

THEOREM 1.4. Let $(M,g) = (S^1, dx \otimes dx)$, let (N,J,h) be a Riemann surface with bounded geometry, and let $k \geq 2$ be an integer. The flow equation (4) with $u_0 \in W^{k,2}(S^1,N)$ admits a unique solution $u \in C^0(\mathbb{R},W^{k,2}(S^1,N))$. In particular u is smooth if u_0 is.

REMARK 1.5. From the physical point of view, the Schrödinger map flow may also be introduced as a generalization of the Heisenberg model for a ferromagnetic spin system. The classical model for this physical system precisely corresponds to maps from the standard circle into $N = S^2$ with the standard metric and complex structure [LL] (for some background see, e.g., [D, DW1, M1, SSB]). Perhaps the most physically natural generalization of the classical model would be to vary the metric on the target S^2 , however it seems that even for small perturbations of the round metric on S^2 global well-posedness was not known before. Theorem 1.4 establishes the global well-posedness of the Cauchy problem describing this physical model when the metric on S^2 is arbitrary.

Let us now outline the key ideas of the proofs. According to Theorem 1.1 we have existence of a local (in time) solution. The strategy of the proof is: First, we translate the flow equation into a system of nonlinear Schrödinger (NLS) equations. Second, for this system of equations we obtain an a priori estimate (and hence local well-posedness) in a weaker norm than that in Theorem 1.1, namely in an appropriate Strichartz norm for $M=\mathbb{R}$ and in L^4 for $M=S^1$. These estimates are crucial since they only depend on the initial energy (that is a conserved quantity) and that in a manner that can be readily converted into a global a priori estimate (and hence global well-posedness for the system of NLS equations) in the same space. Taking derivatives of the flow equation we obtain global a priori estimates in stronger norms and these in turn may be converted back to imply global well-posedness for our original Cauchy problem in $W^{k,2}$ for all $k \geq 2$. While the proofs of both Theorem 1.3 and Theorem 1.4 follow the same general scheme, nevertheless there are substantial differences between the two as we now explain.

First, we consider the case of the real line. This case is considerably simpler due to the non-compactness and dispersiveness. Here we follow Chang-Shatah-Uhlenbeck and write the flow equation in terms of a parallel frame observing the Kähler condition allows one to readily generalize their computations from the Riemann surface case to higher dimensions. The flow equation then reduces to a system of NLS equations. The same Strichartz type calculations as in their study of the Riemann surface case then apply.

Second, we treat the case $M=S^1$. This case is considerably more difficult and is the main contribution of this article. There are two main difficulties. First, using a parallel frame introduces holonomy and so the resulting NLS equations live on \mathbb{R} , the universal cover of S^1 , instead of on S^1 itself. To overcome this we use a certain space-time transformation in order to obtain a system of NLS equations on S^1 in terms of the holonomy representation of N. In addition we need to estimate the variation of the holonomy along the flow. Second, since S^1 is compact the equations are no longer dispersive. To overcome that we adapt Bourgain's results on the cubic NLS to our setting in order to prove local well-posedness in L^4 that depends in such a way on the initial data that it may be used to obtain global well-posedness in the same space.

The article is organized as follows. In Section 2 we treat the case of the real line. In Section 3 we treat the case of maps from the circle into a Riemann surface. Finally, in Section 4 we discuss some of the difficulties that arise when trying to apply our approach to treat maps from the circle into higher dimensional Kähler manifolds. It is conceivable that some of these ideas might be related to showing finite time blow-up for higher-dimensional domains.

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2. Maps of the real line into a Kähler manifold

In this section we consider maps from the Euclidean real line into a complete Kähler manifold (N, J, h) of complex dimension n.

The Schrödinger equation (4) becomes

$$J\nabla_t u - \nabla_x \nabla_x u = 0, \quad u(0) = u_0, \tag{6}$$

$$E(u_0) = \frac{1}{2} \int_{\mathbb{R}} |du_0|^2_{\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \otimes u^* h} dx < \infty, \tag{7}$$

where we use the abbreviated notations $\nabla_t = \nabla_{u_\star \frac{\partial}{\partial t}}$, $\nabla_x = \nabla_{u_\star \frac{\partial}{\partial x}}$ and $\nabla_t u = u_\star \frac{\partial}{\partial t} = \frac{\partial u}{\partial t}$, $\nabla_x u = u_\star \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}$ and denote derivatives of a function f by $f_{,x}$ and $f_{,t}$. The key idea in this section, going back to Chang, Shatah and Uhlenbeck [CSU], is to rewrite (6) in an appropriate frame along the image in such a way that (6) reduces to a system of nonlinear Schrödinger (NLS) equations. In fact our proof closely follows their approach for the Riemann Surface case observing that it readily generalizes to Kähler targets of arbitrary dimension.

Now assume that $u: I \times \mathbb{R} \to N$ is a solution of (6) with I a neighborhood of 0 in \mathbb{R} (given, for example, by Theorem 1.1). Choose an orthonormal frame $\{e_1, \ldots, e_{2n}\}$ for u^*TN with respect to h. We further reduce the structure group to $U(n) \subseteq O(2n)$ by letting $e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$. We identify U(n) with its image in O(2n) under the map $\iota: GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$ given by

$$\iota(A + \sqrt{-1}B) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Note that if $v = x + \sqrt{-1}y \in \mathbb{C}^n$ and

$$\iota(v) = \begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$\iota(Av) = \iota(A)\iota(v).$$

We will use this identification frequently, sometimes omitting the reference to the map ι . In the following we let latin and greek indices take values in $\{1, \ldots, 2n\}$ and $\{1, \ldots, n\}$, respectively. For both alphabets we use the notation

$$\overline{\cdot} = \cdot + n - 1 \pmod{2n} + 1.$$

Therefore barred greek indices take values in $\{n+1,\ldots,2n\}$. Put $e_{\bar{j}}=Je_{j}$ so $e_{\bar{\alpha}}=Je_{\alpha},\ e_{\overline{\alpha+n}}=-e_{\alpha}$. We let, for example, $L^{p}(\mathbb{R}_{x})$ and $L^{p}(\mathbb{R}_{t})$ denote the spaces $L^{p}(\mathbb{R},dx)$ and $L^{p}(\mathbb{R},dt)$, respectively. Finally, given a map $u:(M,g)\to(N,h)$ by $u\in W^{k,p}(M,N)$ we will mean that $\sum_{j=0}^{k-1}\left|\left|\left|\nabla^{j}du\right|\right|\right|_{L^{p}}<\infty$. For example, in this notation we have $E(u)=\left|\left|u\right|\right|_{W^{1,2}(M,N)}^{2}$.

Now we may view the flow equation (6) in this frame. Write for each $(t, x) \in I \times \mathbb{R}$

$$\nabla_x u = \sum_{j=1}^{2n} h(\nabla_x u, e_j) e_j =: a^j e_j, \qquad \nabla_t u = \sum_{j=1}^{2n} h(\nabla_t u, e_j) e_j =: b^j e_j, \qquad (8)$$

where we make use of the Einstein summation convention, namely the appearance of an index both as a subscript and a superscript indicates summation.

Equation (6) can be rewritten as

$$b^{j}e_{\bar{j}} - a_{x}^{j}e_{j} - a^{j}\nabla_{x}e_{j} = 0.$$
(9)

The conservation of energy (see (5)) is expressed as

$$E(u(t)) = E(u_0) = \frac{1}{2} \int_{\mathbb{R}} \sum_{l=1}^{n} (a_l)^2 dx = ||a(t)||_{L^2(\mathbb{R}_x)}^2, \quad \forall t \in \mathbb{R}.$$
 (10)

Note that

$$0 = u_{\star} \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right] = \left[\nabla_t u, \nabla_x u \right]. \tag{11}$$

Since $\nabla J = 0$, differentiating (6) in space yields

$$J\nabla_x\nabla_t u - \nabla_x\nabla_x\nabla_x u = 0,$$

which becomes, using (11) and (8),

$$J(a_{t}^{j}e_{i} + a^{j}\nabla_{t}e_{i}) - a_{xx}^{j}e_{i} - 2a_{x}^{j}\nabla_{x}e_{i} - a^{j}\nabla_{x}\nabla_{x}e_{i} = 0.$$
 (12)

We impose the gauge-fixing condition

$$\nabla_x e_j = 0, \quad j = 1, \dots, 2n. \tag{13}$$

The resulting frame along the image is still unitary since the complex structure commutes with parallel transport. Equation (9) becomes

$$b^j = a^{\bar{j}}_{r}. \tag{14}$$

Note that $u^*TN \to \mathbb{R}$ is trivial and that (13) amounts to fixing a trivializing parallel frame. With this choice the flow on u^*TN is given by

$$a_{,t}^{j}e_{\bar{j}} - a_{,xx}^{j}e_{j} = -a^{j}\nabla_{t}e_{\bar{j}}.$$

$$\tag{15}$$

Along the image, using (13) and (14), and letting R denote the curvature tensor of (N, h), we have

$$\nabla_{x}\nabla_{t}e_{\bar{j}} = R(\nabla_{x}u, \nabla_{t}u)e_{\bar{j}} = a^{k}b^{l}R_{kl\bar{j}}^{\phantom{kl\bar{j}}q}e_{q}$$

$$= (a^{\alpha}b^{\beta}R_{\alpha\beta\bar{j}}^{q} + a^{\bar{\alpha}}b^{\beta}R_{\bar{\alpha}\beta\bar{j}}^{q} + a^{\alpha}b^{\bar{\beta}}R_{\alpha\bar{\beta}\bar{j}}^{q} + a^{\bar{\alpha}}b^{\bar{\beta}}R_{\bar{\alpha}\bar{\beta}\bar{j}}^{q})e_{q}$$

$$= (a^{\alpha}a_{,x}^{\bar{\beta}}R_{\alpha\beta\bar{j}}^{q} + a^{\bar{\alpha}}a_{,x}^{\bar{\beta}}R_{\bar{\alpha}\beta\bar{j}}^{q} - a^{\alpha}a_{,x}^{\beta}R_{\alpha\bar{\beta}\bar{j}}^{q} - a^{\bar{\alpha}}a_{,x}^{\beta}R_{\bar{\alpha}\bar{\beta}\bar{j}}^{q})e_{q}$$

$$= \sum_{\alpha,\beta} \left[(a^{\alpha}a^{\bar{\beta}})_{,x}R_{\alpha\beta\bar{j}}^{q} + \frac{1}{2} [(a^{\bar{\alpha}}a^{\bar{\beta}})_{,x} + (a^{\alpha}a^{\beta})_{,x}]R_{\bar{\alpha}\beta\bar{j}}^{q} \right]e_{q},$$
(16)

where we have used the Kähler condition once more:

$$R_{\alpha\beta\bar{j}}^{\quad q}=R_{\bar{\alpha}\bar{\beta}\bar{j}}^{\quad q},\ R_{\bar{\alpha}\beta\bar{j}}^{\quad q}=-R_{\alpha\bar{\beta}\bar{j}}^{\quad q}.$$

Equation (7) implies that $\lim_{x\to\pm\infty} a^i(t,x)=0$. Therefore, since

$$\nabla_x h(\nabla_t e_{\bar{j}}, e_q) = h(\nabla_x \nabla_t e_{\bar{j}}, e_q),$$

we have

$$\nabla_t e_{\bar{i}}(t,x) =$$

$$\sum_{q=1}^{2n} h(\nabla_{t}e_{\bar{j}}, e_{q})(t, -\infty)e_{q}(t, x)$$

$$+ \Big[\sum_{\alpha, \beta} \Big[a^{\alpha}a^{\bar{\beta}}R_{\alpha\beta\bar{j}}^{\quad q} + \frac{1}{2}[a^{\bar{\alpha}}a^{\bar{\beta}} + a^{\alpha}a^{\beta}]R_{\bar{\alpha}\beta\bar{j}}^{\quad q}](t, x)$$

$$- \int_{(-\infty, x]} \sum_{\alpha, \beta} \Big[a^{\alpha}a^{\bar{\beta}}R_{\alpha\beta\bar{j}}^{\quad q}, x + \frac{1}{2}(a^{\bar{\alpha}}a^{\bar{\beta}} + a^{\alpha}a^{\beta})R_{\bar{\alpha}\beta\bar{j}}^{\quad q}, x\Big](t, y)dy\Big]e_{q}(t, x)$$

$$=: A_{\bar{j}}^{q}(t, -\infty)e_{q}(t, x) + [P_{\bar{j}}^{q}(t, x) + Q_{\bar{j}}^{q}(t, x)]e_{q}(t, x).$$

$$(17)$$

Note that by using (9) we have

$$\nabla_t e_{\bar{j}} = b^k \Gamma^p_{k\bar{j}} e_p = a^{\bar{k}}_{,x} \Gamma^p_{k\bar{j}} e_p. \tag{18}$$

Hence, $h(\nabla_t e_{\bar{j}}, e_q) = a_{,x}^{\bar{k}} \Gamma_{k\bar{j}}^q$ and so we may assume that $A_{\bar{j}}^q$ vanishes at $(t, -\infty)$. To justify this note that this is indeed the case for the local solution of our equation given by Theorem 1.1; even though this assumption makes use of the finiteness of the $W^{2,2}$ norm of that local solution, the important point is that eventually our estimates will not depend on the $W^{2,2}$ norm of u (equivalently on the $W^{1,2}$ norm of a), and so the proof of the a priori estimate for the system of NLS equations (21) below (for a) goes through, with this assumption.

Note that $R_{klp}^{q}_{,x} = a^{s}R_{klp}^{q}_{,s}$. Therefore $|P_{j}^{q}| < C||\mathbf{a}||^{2}$ and $|Q_{j}^{q}| < C\int_{M} ||\mathbf{a}||^{3}dx$. Finally (15) transforms to the following system of NLS equations

$$-a_{,t}^{\bar{\gamma}} - a_{,xx}^{\gamma} = -a^j P_j^{\gamma} - a^j Q_j^{\gamma}, \qquad \gamma = 1, \dots, n,$$

$$(19)$$

$$a_{,t}^{\gamma} - a_{,xx}^{\bar{\gamma}} = -a^j P_j^{\bar{\gamma}} - a^j Q_j^{\bar{\gamma}}, \qquad \gamma = 1, \dots, n,$$

$$(20)$$

or, letting $J_0 = \iota(\sqrt{-1}I)$,

$$J_0 \mathbf{a}_{.t} = \mathbf{a}_{.xx} - \mathbf{P} \cdot \mathbf{a} - \mathbf{Q} \cdot \mathbf{a}. \tag{21}$$

where $\mathbf{a} = (a^1, \dots, a^{2n})^T$, $\mathbf{P} = (P_j^k)$, $\mathbf{Q} = (Q_j^k)$.

Equivalently, using the aformentioned identification ι of $GL(n,\mathbb{C})$ with a subset of $GL(2n,\mathbb{R})$,

$$\sqrt{-1}\mathbf{\Phi}_{,t} = \mathbf{\Phi}_{,xx} - \mathbf{S} \cdot \mathbf{\Phi} - \mathbf{T} \cdot \mathbf{\Phi},\tag{22}$$

where $\mathbf{\Phi} = \iota^{-1}(\mathbf{a}) = (a^1 + \sqrt{-1}a^{\bar{1}}, \dots, a^n + \sqrt{-1}a^{\bar{n}})^T$, $\mathbf{S} = (S_{\alpha}^{\beta}) = \iota^{-1}(\mathbf{P})$, $\mathbf{T} = (T_{\alpha}^{\beta}) = \iota^{-1}(\mathbf{Q})$ and $|S_{\alpha}^{\beta}| < C||\mathbf{\Phi}||^2$ and $|T_{\alpha}^{\beta}| < C\int_M ||\mathbf{\Phi}||^3 dx$. Here C depends only on the geometry of (N, J, h), which we assume to be bounded.

REMARK **2.1.** In the case of a variable complete smooth metric $(M, g) = (\mathbb{R}, \alpha^{-1} dx \otimes dx)$ with $\alpha > 0$ equation (4) becomes

$$b^k e_{\bar{k}} = \alpha a_{,x}^k e_k + \frac{1}{2} \alpha_{,x} a^k e_k, \tag{23}$$

which can then be transformed, as before, to

$$J_0 \mathbf{a}_{,t} = \alpha \mathbf{a}_{,xx} + \frac{3\alpha_{,x}}{2} \mathbf{a}_{,x} + \frac{\alpha_{,xx}}{2} \mathbf{a} - \mathbf{P} \cdot \mathbf{a} - \mathbf{Q} \cdot \mathbf{a}. \tag{24}$$

Equivalently, again using the map $\iota: GL(n,\mathbb{C}) \to GL(2n,\mathbb{R})$,

$$\sqrt{-1}\mathbf{\Phi}_{,t} = \alpha\mathbf{\Phi}_{,xx} + \frac{3\alpha_{,x}}{2}\mathbf{\Phi}_{,x} + \frac{\alpha_{,xx}}{2}\mathbf{\Phi} - \mathbf{S} \cdot \mathbf{\Phi} - \mathbf{T} \cdot \mathbf{\Phi}. \tag{25}$$

The only obstacle to treating this equation using the methods below is the first derivative term on the right hand side.

Therefore we have reduced the original flow equation for the map to a system of NLS equations for the frame coefficients of the gradient of the map.

We are now in a position to demonstrate the following theorem that is the main result of this section.

THEOREM 2.2. Let $(M,g)=(\mathbb{R}, dx\otimes dx)$ and let (N,J,h) be a complete Kähler manifold with bounded geometry. Then for integers $k\geq 2$ equation (4) with $u_0\in W^{k,2}(\mathbb{R},N)$ admits a unique solution $u\in C^0(\mathbb{R},W^{k,2}(\mathbb{R},N))$. In particular u is smooth if u_0 is.

Proof. First, we have by Theorem 1.1 local well-posedness of the original Schrödinger map flow (6) in $C^0(\mathbb{R}_t, W^{k,2}(\mathbb{R}, N))$ for $k \geq 2$. Our purpose is now to prove a local a priori estimate in a space that is weaker than $W^{2,2}(\mathbb{R}, N)$. Equivalently, we will prove an estimate for the frame coefficients a in a norm weaker than $W^{1,2}(\mathbb{R}, \mathbb{R}^{2n})$. More precisely, we first assume we are given a smooth initial data and prove estimates on the smooth local solution given by Theorem 1.1 in the weaker space $L^4(\mathbb{R}_{t,loc}, W^{1,\infty}(\mathbb{R}, N)) \cap C^0(\mathbb{R}_t, W^{1,2}(\mathbb{R}, N))$ that will be independent of this smoothness assumption. The key will be that these estimates will depend only on the initial energy $E(u_0)$. Since the energy is conserved these will then lead to a global a priori estimate in the same space. Then, by taking derivatives of the flow equation we will obtain the global well-posedness claimed in the statement.

Each of the equations in the system (22) is of exactly the same type as the equation obtained by Chang-Shatah-Uhlenbeck [CSU] for the case of a target Riemann surface. The only difference is that each Φ^j depends also on the other Φ^k , k = 1, ..., n. However this dependence is only in the nonlinear terms and not in the terms involving derivatives. It therefore follows that their proof of local well-posedness in the Strichartz spaces $L^4(\mathbb{R}_t, L^{\infty}(\mathbb{R}_x)) \cap C^0(\mathbb{R}_t, L^2(\mathbb{R}_x))$ (see (28) below for notation) carries over to our system of NLS equations. Since this estimate depends only on the initial energy (that is a conserved quantity) we obtain global well-posedness in $L^4(\mathbb{R}_t, L^{\infty}(\mathbb{R}_x))$. Taking a derivative of the equation and working in the intersection of the Strichartz spaces $L^4(\mathbb{R}_t, L^{1,\infty}(\mathbb{R}_x))$ and $L^{\infty}(\mathbb{R}_t, W^{1,2}(\mathbb{R}_x))$ as in their original proof then yields global well-posedness in $W^{2,2}(\mathbb{R}, N)$ for the original flow equation (i.e., global existence in time for initial data in $W^{2,2}(\mathbb{R}, N)$). One may then show using similar computations global well-posedness in $W^{k,2}$ for each $k \geq 2$.

In fact, although we will not carry this out here, one may prove local (and hence global) well-posedness for (22) in other Strichartz spaces (these are by definition the spaces $L^q(\mathbb{R}_t, L^r(\mathbb{R}_x))$ specified by Lemma 2.3 below) as well, e.g., $L^6(\mathbb{R} \times \mathbb{R})$.

For the benefit of the reader that may not be familiar with standard Strichartz estimate techniques we include here the detailed proof of the Chang-Shatah-Uhlenbeck $L^4(\mathbb{R}_t, L^\infty(\mathbb{R}_x))$ estimate. No originality is claimed here. This also serves to provide some perspective on the differences between this case and the case of the circle, treated in the following sections. The estimates on higher derivatives are similar and for that we refer to their original article.

Suppose a function $c: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies the NLS equation

$$\sqrt{-1} c_{,t} = c_{,xx} + F, \quad \forall t \in [0, T], \qquad c(0) = f,$$
(26)

for some function $F:[0,T]\times\mathbb{R}\to\mathbb{C}$ that may depend on c nonlinearly (but not on its derivatives). One then has the integral expression (Duhamel formula)

$$c(t,x) = \int_{\mathbb{R}} f(y) \frac{e^{-\sqrt{-1}|x-y|^2/4t}}{\sqrt{2\pi t}} dy - \sqrt{-1} \int_0^t \int_{\mathbb{R}} F(s,y) \frac{e^{-\sqrt{-1}|x-y|^2/4(t-s)}}{\sqrt{2\pi (t-s)}} dy \wedge ds.$$
(27)

Denote the Schrödinger operator by

$$S(t)f := \int_{\mathbb{R}} f(y) \frac{e^{-\sqrt{-1}|x-y|^2/4t}}{\sqrt{2\pi t}} dy.$$

We now recall the Strichartz estimates (on \mathbb{R}). For appropriate q, r we denote by $L^q(\mathbb{R}, L^r(\mathbb{R}))$ the Banach space equipped with the norm

$$||f||_{L^q(\mathbb{R},L^r(\mathbb{R}))} := ||||f||_{L^r(\mathbb{R}_x)}||_{L^q(\mathbb{R}_t)}.$$
 (28)

LEMMA 2.3. (See [C], p. 33.) Let q, r satisfy $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, with $r \in [2, \infty]$ and let $f \in L^2(\mathbb{R}_x)$. Then the function $t \mapsto S(t)f$ belongs to $L^q(\mathbb{R}_t, L^r(\mathbb{R}_x)) \cap C^0(\mathbb{R}_t, L^2(\mathbb{R}_x))$ and there is a constant independent of (q, r) and $f \in L^2(\mathbb{R})$ such that

$$||S(\cdot)f||_{L^q(\mathbb{R}_t,L^r(\mathbb{R}_x))} \le C||f||_{L^2(\mathbb{R})}.$$
(29)

In our situation we know that $||\Phi||_{L^2}$ is constant in time (recall (10)). Assume also that $\Phi \in L^4([0,T], L^{\infty}(\mathbb{R}_x))$. We will now show that in fact the $L^4([0,T], L^{\infty}(\mathbb{R}_x))$ norm of Φ is controlled by its L^2 norm and the geometry. This will imply local and eventually global well-posedness in $L^4(\mathbb{R}_t, L^{\infty}(\mathbb{R}_x))$.

Let
$$F = -\mathbf{S} \cdot \mathbf{\Phi} - \mathbf{T} \cdot \mathbf{\Phi}$$
. Then

$$\mathbf{\Phi}^{j}(t) = S(t - t_{1})\mathbf{\Phi}^{j}(t_{1}) - \sqrt{-1} \int_{t_{1}}^{t} S(t - s)F(s, \cdot)ds.$$
(30)

The first term of (30) is in $L^4([t_1,t],L^{\infty}(\mathbb{R}))$ by the Strichartz estimate (29). We will now show that the second term is also in this space.

First, we consider the term $\mathbf{S} \cdot \mathbf{\Phi} \leq C||\mathbf{\Phi}||^3$. By applying the Strichartz estimate under the integral sign and using energy conservation we obtain

$$\left|\left|\int_{t_{1}}^{t} S(t-s)(\mathbf{S} \cdot \mathbf{\Phi})(s,\cdot)ds\right|\right|_{L^{4}([t_{1},t_{2}],L^{\infty}(\mathbb{R}))}$$

$$\leq \int_{t_{1}}^{t} \left|\left|S(t-s)(\mathbf{S} \cdot \mathbf{\Phi})(s,\cdot)\right|\right|_{L^{4}([t_{1},t_{2}],L^{\infty}(\mathbb{R}))}ds$$

$$\leq C \int_{t_{1}}^{t} \left|\left|(\mathbf{S} \cdot \mathbf{\Phi})(s,\cdot)\right|\right|_{L^{2}(\mathbb{R})}ds$$

$$\leq C' \int_{t_{1}}^{t} \left|\left|\left|\mathbf{\Phi}(s,\cdot)\right|^{3}\right|\right|_{L^{2}(\mathbb{R})}ds$$

$$\leq C' \int_{t_{1}}^{t} \left|\left|\mathbf{\Phi}(s,\cdot)\right|\right|_{L^{\infty}(\mathbb{R})}^{2} \left|\left|\mathbf{\Phi}(s,\cdot)\right|\right|_{L^{2}(\mathbb{R})}ds$$

$$\leq C'' \int_{t_{1}}^{t} \left|\left|\mathbf{\Phi}(s,\cdot)\right|\right|_{L^{\infty}(\mathbb{R})}^{2} \left|\left|\mathbf{\Phi}(s,\cdot)\right|\right|_{L^{2}(\mathbb{R})}ds$$

$$\leq C'' \int_{t_{1}}^{t} \left|\left|\mathbf{\Phi}(s,\cdot)\right|\right|_{L^{\infty}(\mathbb{R})}^{2} ds \leq C'' |t-t_{1}|^{1/2} \left|\left|\mathbf{\Phi}\right|\right|_{L^{4}([t_{1},t_{2}],L^{\infty}(\mathbb{R}))}^{2}.$$

Second, we consider the term $\mathbf{T} \cdot \mathbf{\Phi} \leq C||\mathbf{\Phi}|| \int_{\mathbb{R}} ||\mathbf{\Phi}||^3 dx$. Again, by applying the Strichartz estimate under the integral sign and using energy conservation we obtain

$$\left| \left| \int_{t_{1}}^{t} S(t-s)(\mathbf{T} \cdot \mathbf{\Phi})(s, \cdot) ds \right| \right|_{L^{4}([t_{1}, t], L^{\infty}(\mathbb{R}))}$$

$$\leq C' \int_{t_{1}}^{t} \left| \left| \mathbf{\Phi} \right| \left| \left| \mathbf{\Phi} \right|^{3} \right| \right|_{L^{1}(\mathbb{R})} \right| \right|_{L^{2}(\mathbb{R})} ds.$$

$$\leq C' \int_{t_{1}}^{t} \left| \left| \mathbf{\Phi} \right| \left| \mathbf{\Phi} \right| \right|_{L^{\infty}(\mathbb{R})} \left| \left| \mathbf{\Phi} \right| \right|_{L^{2}(\mathbb{R})} \right| \right|_{L^{2}(\mathbb{R})} ds.$$

$$\leq C'' \int_{t_{1}}^{t} \left| \left| \mathbf{\Phi} \right| \right|_{L^{\infty}(\mathbb{R})} ds \leq C'' |t - t_{1}|^{3/4} \left| \left| \mathbf{\Phi} \right| \right|_{L^{4}([t_{1}, t], L^{\infty}(\mathbb{R}))}.$$

$$(32)$$

Combining (31) and (32) we thus obtain, by choosing $|t-t_1|$ small enough (depending only on the initial energy), an estimate on $||\mathbf{\Phi}||_{L^4([t_1,t],L^{\infty}(\mathbb{R}))}$, depending only on the geometry of (N,h) and the initial energy. This then implies global well-posedness in $L^4(\mathbb{R}_t,L^{\infty}(\mathbb{R}_x))$, by which we mean that for all T>0 we have

$$\int_0^T ||\mathbf{\Phi}(s,\cdot)||_{L^{\infty}(\mathbb{R}_x)}^4 ds < \infty.$$

As indicated earlier the higher derivatives estimates follow similar computations. This concludes the proof of Theorem 2.2.

REMARK 2.4. It is essential to use Theorem 1.1 since even after we reduce the Schrödinger map flow to a system of NLS equations and after proving that a unique global solution for (22) exists it is not completely obvious how to go from such a solution for $\nabla_x u$ to an actual map u into N.

3. Maps of the circle into a Riemann surface

In this section we consider the Schrödinger map flow with the domain being the round circle. Compared with the previous section, the discussion here is more delicate due to the fact that the domain is no longer simply-connected (introduces holonomy) nor non-compact (lack of dispersion).

Let $u: I \times S^1 \to N$ where $I \subset \mathbb{R}$ is a neighborhood of 0. The bundle $u^*TN \to I \times S^1$ is no longer trivial and so fixing a frame satisfying (13) does not yield a trivialization. To describe the solution of (13) we work instead with \mathbb{Z} -invariant objects over \mathbb{R} . We therefore make the identifications

$$\operatorname{Maps}(S^1, N) \cong \operatorname{Maps}(\mathbb{R}, N)^{\mathbb{Z}}, \quad \Gamma(I \times S^1, u^*TN) \cong \Gamma(I \times \mathbb{R}, u^*TN)^{\mathbb{Z}}, \quad (33)$$

the superscript denoting \mathbb{Z} -invariant objects, and take the freedom to use either one of these identifications interchangibly. Similar identifications will be made for all the other tensor bundles encountered over $I \times S^1$ (e.g., $u^*(T^*N \otimes T^*N \otimes T^*N \otimes TN)$).

Recall that parallel transport is defined as a map $P: u \mapsto \operatorname{Aut}(u(0)^*TN, u(1)^*TN)$ for all $u \in C^{\infty}([0,1],N)$, which on any Kähler manifold restricts to an operator $P: C^{\infty}((S^1,\operatorname{pt}),N) \to \iota(U(n))$ on base-pointed loops. Formally, a solution of (13) is given by $e(t,x) = P(u(t)|_{[0,x]})e(t,0)$. This can be described somewhat more explicitly as follows.

Let U denote a contractible open set in N and and let $e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$ denote a local orthonormal frame. Assume $u: I \times \mathbb{R} \to N$ is a solution of (4), a collection of loops in N which we will initially assume to be contained in U (and so, in effect, these loops are all contractible in N). Along the image of our flow we denote by $\alpha^1, \ldots, \alpha^{2n}$ the dual 1-forms to e_1, \ldots, e_{2n} . The Levi-Civita connection along our flow restricted to this patch is represented by a section $A_U = \Gamma^k_{ij}\alpha^i$ of $T^*N|_U \otimes u(n)$ which pulls back to a connection form $u^*A_U = \Gamma^k_{ij}a^idx =: B_Udx$ for the pulled-back bundle. A section $e = E^j e_j$ of the pulled-back bundle (as in (33)) is then (locally) parallel when

$$0 = \nabla e = \frac{\partial E^j}{\partial x} e_j \otimes dx + B_U \cdot e \otimes dx = (E^j_{,x} + B_U{}^j_k E^k) e_j \otimes dx.$$
 (34)

The solution of this first-order matrix equation simplifies considerably in the case n = 1. The matrices B_U then lie in the trivial Lie algebra $so(2) \cong u(1)$ and so their exponentials commute. One may therefore integrate (34) to obtain

$$e(t,1) = \exp(-\int_0^1 B_U dx) e(t,0). \tag{35}$$

If D_u is the disc bounded by u and contained in U, and K denotes the Gaussian curvature of N, then Stokes' Theorem gives

$$e(t,1) = \exp(-\int_{D_u} dA_U)e(t,0) = \exp(-\int_{D_u} KdV_{N,h})e(t,0)$$

(possibly up to a factor of 2π , depending on conventions) from which it becomes evident that one may relax the assumption above (for the moment still restricting to contractible loops) and work globally (one might have two choices for D_u then). Also, we see that the holonomy factor is independent of the starting point on the loop. In fact this last fact is seen to be true also for non-contractible loops. We have therefore a well-defined holonomy map

$$P: C^{\infty}((S^1, pt), N) \to SO(2) = \iota(U(1)).$$

Next, for general u, since $u(0, S^1)$ and $u(t, S^1)$ are homotopic for any $t \in I$ we may define the surface $D_u = u([0, t] \times S^1)$ and as chains on $N \partial D_u = u(t, S^1) - u(0, S^1)$. Let K denote the Gaussian curvature of (N, h). Then we have once again by Stokes' Theorem

$$e(t,1) = P(u)e(t,0) = \exp(-\int_{D_u} KdV_{N,h})P(u_0)e(t,0),$$

or for any $x \in \mathbb{R}$ and $l \in \mathbb{N}$

$$e(t, x+l) = P(u)^{l} e(t, x) = \exp(-l \int_{D_u} K dV_{N,h}) P(u_0)^{l} e(t, 0).$$
 (36)

Therefore a solution of (13) produces a parallel section of $\Gamma(\mathbb{R}, u^*TN)$ rather than of $\Gamma(\mathbb{R}, u^*TN)^{\mathbb{Z}}$. In expressing our \mathbb{Z} -invariant tensors in terms of the frame $\{e_j\}_{j=1}^{2n}$ we therefore use coefficients satisfying a relation appropriately proportional to (36). For example if $v \in \Gamma(\mathbb{R}, u^*TN)^{\mathbb{Z}}$ then we may write $v = v^j e_j$ with $v^j(x+l) = P(u)^{-l}v^j(x)$ (while on the other hand sections of endomorphism tensor bundles require no adjustment when n=1).

Going through the computations of §2 it follows that Equation (15) still holds. We then obtain

$$\nabla_t e_{\bar{i}}(t,x) =$$

The terms depending on the fixed point x_0 are in a sense worse than those that depend on the variable point x since the former must be evaluated in $L^{\infty}(\mathbb{R}_x)$ norm. To overcome this apparent obstacle we average over S^1 (namely, x_0 in the range (x-1,x)) to obtain

$$\nabla_t e_{\bar{j}}(t,x) =$$

$$\sum_{q=1}^{2n} \left(\int_{S^{1}} h(\nabla_{t}e_{\bar{j}}, e_{q})(t, x_{0}) dx_{0} \right) e_{q}(t, x)
+ \left[\sum_{\alpha, \beta} \left(\left[a^{\alpha}a^{\bar{\beta}}R_{\alpha\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}} + \frac{1}{2} (a^{\bar{\alpha}}a^{\bar{\beta}} + a^{\alpha}a^{\beta})R_{\bar{\alpha}\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}}^{\phantom{\alpha\beta\bar{j}}} \right](t, x) \right]
- \int_{S^{1}} \left[a^{\alpha}a^{\bar{\beta}}R_{\alpha\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}} + \frac{1}{2} (a^{\bar{\alpha}}a^{\bar{\beta}} + a^{\alpha}a^{\beta})R_{\bar{\alpha}\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}} \right](t, x_{0}) dx_{0} \right)
- \int_{[x_{0}, x]} \sum_{\alpha, \beta} \left[a^{\alpha}a^{\bar{\beta}}a^{s}R_{\alpha\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}} \right]_{s}^{q} + \frac{1}{2} (a^{\bar{\alpha}}a^{\bar{\beta}} + a^{\alpha}a^{\beta}a^{s})R_{\bar{\alpha}\beta\bar{j}}^{\phantom{\alpha\beta\bar{j}}} \right](t, y) dy e_{q}(t, x)
=: \left(T_{\bar{j}}^{q} + P_{\bar{j}}^{q} - \int_{S^{1}} P_{\bar{j}}^{q}(t, x_{0}) dx_{0} + Q_{\bar{j}}^{q} \right) e_{q}. \tag{38}$$

Note that according to (18) we have $\nabla_t e_{\bar{j}} = a^{\bar{k}}_{,x} \Gamma^p_{k\bar{j}} e_p$, hence

$$h(\nabla_t e_{\bar{i}}, e_q) = a_{.x}^{\bar{k}} \Gamma_{k\bar{i}}^q. \tag{39}$$

Note that in (39) the left hand side, hence also the right hand side, are bona fide functions on S^1 (even though each term separately in the product on the right hand

side is not). Therefore,

$$\int_{S^1} h(\nabla_t e_{\bar{j}}, e_q)(t, x_0) dx_0 = \int_{S^1} a_{,x}^{\bar{k}} \Gamma_{k\bar{j}}^q dx_0 = -\int_{S^1} a^{\bar{k}} \Gamma_{k\bar{j},x}^q dx_0 = -\int_{S^1} a^{\bar{k}} a^p \Gamma_{k\bar{j},p}^q dx_0.$$

Switching to complex notation, as in (22), we have

$$\sqrt{-1}\Phi_{,t} = \Phi_{,xx} - \mathbf{Q} \cdot \mathbf{\Phi} - \mathbf{S} \cdot \mathbf{\Phi} + \mathbf{W} \cdot \mathbf{\Phi} - \mathbf{T} \cdot \mathbf{\Phi}, \tag{40}$$

$$\mathbf{\Phi}^{\alpha}(t, x+l) = P(u(t)|_{[0,1]})^{-l} \mathbf{\Phi}^{\alpha}(t, x), \tag{41}$$

where $\mathbf{Q} := \iota^{-1}(T^q_{\bar{j}}), \ \mathbf{S} := \iota^{-1}(P^q_{\bar{j}}), \ \mathbf{T} := \iota^{-1}(Q^q_{\bar{j}}), \ \mathbf{W} := \iota^{-1}(\int_{S^1} P^q_{\bar{j}}(t, x_0) dx_0),$ and

$$||\mathbf{Q}|| < C \int_{S^1} ||\mathbf{\Phi}||^2 dx, \quad ||\mathbf{W}|| < C \int_{S^1} ||\mathbf{\Phi}||^2 dx,$$

$$||\mathbf{S}|| < C ||\mathbf{\Phi}||^2 dx, \quad ||\mathbf{T}|| < C \int_{S^1} ||\mathbf{\Phi}||^3 dx,$$
(42)

where C > 0 depends only on the geometry of (N, h). We emphasize that Equations (40)-(41) are on $I \times \mathbb{R}$. Put

$$P(u(t)|_{[0,1]}) =: e^{\sqrt{-1}\theta(t)} \in U(1), \qquad \theta \in \mathbb{R},$$

and set

$$a := \mathbf{\Phi}^1 = a^1 + \sqrt{-1}a^{\bar{1}}$$

(note that we cannot restrict θ to $[-\pi,\pi)$ in order not to violate continuity of θ). Note that following an earlier remark $\mathbf{Q}=Q_1^1,\mathbf{S}=S_1^1,\mathbf{W}=W_1^1,\mathbf{T}=T_1^1$ are \mathbb{Z} -invariant. Also

$$\varphi(t,x) := e^{\sqrt{-1}\theta x} a(t,x) \tag{43}$$

is \mathbb{Z} -invariant. Moreover, so are all of its x-derivatives. To wit,

$$\varphi(t,x)_{,x} = \sqrt{-1}\theta\varphi(t,x) + e^{\sqrt{-1}\theta x}a(t,x)_{,x} = \varphi(t,x+1)_{,x}$$
(44)

since $(e^{\sqrt{-1}\theta}a(t,x+1))_{,x} = a(t,x)_{,x}$, and the claim now follows by induction. It follows that the estimates we will obtain for φ will imply the same estimates for a. After the change of variable (43) Equation (40) becomes

$$\sqrt{-1}\varphi_{,t} = \varphi_{,xx} - 2\sqrt{-1}\theta\varphi_{,x} - (\theta^2 + x\theta_{,t} + Q_1^1 + S_1^1 - W_1^1 + T_1^1)\varphi.$$
 (45)

Let $\beta: I \times \mathbb{R} \to I \times \mathbb{R}$ be given by

$$\beta(t,x) = (t, x - 2 \int_{[0,t]} \theta ds).$$

Let

$$\tilde{x} := x + 2 \int_{[0,t]} \theta ds.$$

Writing

$$(t,x) = \beta(t,x+2\int_{[0,t]}\theta ds) = \beta(t,\tilde{x}),$$

Equation (40) becomes

$$\sqrt{-1}(\varphi \circ \beta)_{,t}(t,\tilde{x}) = (\varphi \circ \beta)_{,\tilde{x}\tilde{x}}(t,\tilde{x}) - \left(\theta^{2}(t) + (\tilde{x} - 2\int_{[0,t]}\theta ds)\theta_{,t}(t) + (Q_{1}^{1} \circ \beta + S_{1}^{1} \circ \beta - W_{1}^{1} \circ \beta + T_{1}^{1} \circ \beta)(t,\tilde{x})\right)(\varphi \circ \beta)(t,\tilde{x}).$$

$$(46)$$

This equation is on $I \times S^1$.

REMARK **3.1.** Note that here it was crucial that θ does not depend on x in order to have $\frac{\partial \tilde{x}}{\partial x} = 1$. This is also the difference from the situation in Equation (25).

The main result of this section is:

THEOREM 3.2. Let $(M,g) = (S^1, dx \otimes dx)$ and let (N,J,h) be a complete Riemann surface with bounded geometry. Then the system of NLS equations (46) is locally well-posed in the space $L^4(\mathbb{R}, L^4(S^1, \mathbb{R}^{2n}))$.

This will then imply:

COROLLARY 3.3. Let $(M,g) = (S^1, dx \otimes dx)$ and let (N, J, h) be a complete Riemann surface with bounded geometry. Then for integers $k \geq 2$ equation (4) with $u_0 \in W^{k,2}(S^1, N)$ admits a unique solution $u \in C^0(\mathbb{R}, W^{k,2}(S^1, N))$. In particular u is smooth if u_0 is.

We turn to the proof of Theorem 3.2.

Proof. We will use Equation (46) in order to obtain a priori estimates on

$$\tilde{\varphi}(t, \tilde{x}) := \varphi \circ \beta(t, \tilde{x}).$$

The estimates on $\tilde{\varphi}$ and on φ are equivalent since the two functions only differ by a time-dependent translation in the space direction. We will localize in time: indeed it is enough to prove existence of local (in time) solutions of Equation (46) in $C^0(\mathbb{R}_{t,loc}, L^2(S^1)) \cap L^4(\mathbb{R}_{t,loc} \times S^1)$ depending in a good manner only on $||\tilde{\varphi}||_{L^2(\mathbb{R}_{\tilde{x}})} = ||a||_{L^2(\mathbb{R}_x)}$ and a bounded constant depending on time, since that will rule out finite-time blow-up.

Let us now recall some of the work of Bourgain that will be of central importance in what follows [B] (see also [G]). First, we recall the following Fourier restriction estimates of Bourgain:

LEMMA **3.4.** (See [B], p. 112.) Let φ be a periodic solution of the linear Schrödinger equation on S^1 . Then

$$||\varphi||_{L^4(S^1\times S^1)} \le \sqrt{2}||\varphi(0)||_{L^2(S^1)},$$

and dually

$$||\varphi||_{L^2(S^1 \times S^1)} \le \sqrt{2} ||\varphi||_{L^{4/3}(S^1 \times S^1)}.$$

More generally, Bourgain proved the following fundamental result that allows for the same estimate—now with appropriate weights—even for an arbitrary function whose Fourier modes are not necessarily restricted to the parabola $\{(n, n^2) : n \in \mathbb{Z}\}$. We state the result although we will only directly use a consequence of it.

LEMMA 3.5. (See [B], Proposition 2.33.) Let $f(x,t) = \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{\sqrt{-1}(mx+nt)}$ be a function on $S^1 \times S^1$. Then

$$\left(\sum_{m,n\in\mathbb{Z}}(|n-m^2|+1)^{-3/4}|a_{m,n}|^2\right)^{1/2}\leq c||f||_{L^{4/3}(S^1\times S^1)}.$$

In addition, if $|\lambda_{m,n}| \leq (1+|n-m^2|)^{-3/4}$, then

$$||\sum_{m,n\in\mathbb{Z}}\lambda_{m,n}a_{m,n}e^{\sqrt{-1}(mx+nt)}||_{L^4(S^1\times S^1)}\leq c||f||_{L^{4/3}(S^1\times S^1)}.$$

In both estimates c > 0 is some universal constant.

Using this estimate Bourgain obtains the following L^4 estimate for the nonlinear contribution in Duhamel's formula. This estimate will play a central rôle below.

LEMMA **3.6.** (See [B], §4) Let $F \in L^{4/3}(S^1 \times S^1)$. For any $0 < \delta < 1/8$ and $0 < B < \frac{1}{100\delta}$ there holds

$$\left| \left| \int_0^{2\delta} S(t-\tau)F(\tau,x)d\tau \right| \right|_{L^4(S^1\times S^1)} \le C(B^{-1/4} + \delta B)||F||_{L^{4/3}(S^1\times S^1)},$$

where C > 0 is some universal constant.

The constant B can be thought of as a Fourier mode cut-off parameter, measuring distance of a lattice point in \mathbb{Z}^2 from the parabola $\{(m, m^2) : m \in \mathbb{Z}\}$. The constant δ is the time cut-off parameter.

Equation (46) is equivalent to the integral equation

$$\tilde{\varphi}(t,\tilde{x}) = S(t)\tilde{\varphi}(0,\tilde{x}) - \sqrt{-1} \int_0^t S(t-\tau)F(\tau,\tilde{x})d\tau. \tag{47}$$

with

$$\begin{split} F(\tau,\tilde{x}) &= -\Big(\theta^2(\tau) + (\tilde{x} - 2\int_{[0,\tau]} \theta \, ds)\theta_{,t}(\tau) \\ &+ (Q_1^1 \circ \beta + S_1^1 \circ \beta - W_1^1 \circ \beta + T_1^1 \circ \beta)(\tau,\tilde{x})\Big)\tilde{\varphi}(\tau,\tilde{x}). \end{split} \tag{48}$$

There is a subtlety here: the time derivative of $\tilde{\varphi}$ (or of φ) is not necessarily \mathbb{Z} -invariant (in \tilde{x}). However, Equation (46) holds on $I \times S^1$ and it is equivalent to the integral equation (47).

We would like to obtain an a priori L^4 estimate on $\tilde{\varphi}$. We localize in time, namely multiply Equation (47) by a smooth cut-off function in time $\psi(t)$ satisfying $\psi=1$ on $[-\delta, \delta]$ and $\psi=0$ for $|t|\geq 2\delta$. Here δ is a positive number smaller than 1/8 to be specified later. We may thus regard $\psi\tilde{\varphi}$ as a function on $S^1\times S^1$ with period 1 in both the t and \tilde{x} variables and Bourgain's estimates apply.

First, the linear term satisfies

$$||\psi S(t)\tilde{\varphi}(0,\tilde{x})||_{L^4(S^1\times S^1)} \le \sqrt{2}||\tilde{\varphi}(0,\cdot)||_{L^2(S^1)} = \sqrt{2E(u_0)},$$

according to Lemma 3.4.

Next, we estimate the integral term. The terms involving \mathbf{Q} and \mathbf{W} are simpler since $||\mathbf{Q}||$ and $||\mathbf{W}||$ are uniformly bounded according to (42) and conservation of energy.

We now turn to the other terms. First, using Lemma 3.4 under the integral sign, and assuming $||\theta||_{L^{\infty}} \leq C$, we have

$$||\psi \int_{0}^{t} S(t-\tau) (\theta^{2}(\tau)\tilde{\varphi}(\tau,\tilde{x})) d\tau||_{L^{4}(S^{1}\times S^{1})} \leq \int_{0}^{2\delta} ||\theta^{2}(\tau)\tilde{\varphi}(\tau,\cdot)||_{L^{2}(S^{1})} d\tau$$

$$\leq 2C^{2}\delta ||\varphi(0)||_{L^{2}(S^{1})}. \tag{49}$$

To show that this assumption holds use the representation of the holonomy given by (35): $|\theta(t)| \leq \int_0^1 |\Gamma_{ij}^k| |a^i| dx \leq C' E(u_0)^{1/2}$ where we have used the assumption of bounded geometry—indeed it implies that Christoffel symbols are uniformly bounded [E].

Second, $|\tilde{x}| \leq 1$ and so $|\tilde{x} - 2\int_{[0,\tau]} \theta ds| \leq 1 + 2 \cdot 1 \cdot C$. Let $\{\alpha_1, \alpha_{\bar{1}}\}$ be an orthonormal coframe dual to $\{e_1, e_{\bar{1}}\}$. To compute the time derivative of θ , recall that by Equation (36) we have

$$\theta(t) = \int_{D_u} K dV_{N,h} = \int_{D_u} K \alpha_1 \wedge \alpha_{\bar{1}} = \int_{I \times S^1} K \circ u(t,x) [a^1 b^{\bar{1}} - a^{\bar{1}} b^1] dx \wedge dt,$$

since $u^*\alpha_1 = a^1 dx + b^1 dt$, $u^*\alpha_{\bar{1}} = a^{\bar{1}} dx + b^{\bar{1}} dt$. Combining this with the fact that by (9) we have $b^k = a^{\bar{k}}_{,x}$, we then have

$$\theta_{,t} = \int_{S^1} K \circ u(t,x) (a^1 b^{\bar{1}} - a^{\bar{1}} b^1) dx = -\frac{1}{2} \int_{S^1} K \circ u(t,x) ((a^1)^2 + (a^{\bar{1}})^2)_{,x} dx.$$

Integrating by parts this becomes

$$\theta_{,t} = \frac{1}{2} \int_{S^1} (K \circ u(t,x))_{,x} ((a^1)^2 + (a^{\bar{1}})^2) dx = \frac{1}{2} \int_{S^1} K_{,s} \circ u(t,x) a^s ((a^1)^2 + (a^{\bar{1}})^2) dx.$$

By bounded geometry we therefore have

$$||\theta_{t}||_{L^{\infty}} \le C||a||_{L^{3}(S^{1})}^{3}. \tag{50}$$

Therefore the term $(\tilde{x}-2\int_{[0,\tau]}\theta ds)\theta_{,t}(\tau)\tilde{\varphi}(\tau,\tilde{x})$ behaves in the same way as the term $T_1^1\circ\beta\tilde{\varphi}(\tau,\tilde{x})$ in (48), and so it's enough to treat the latter. We will do that shortly. Third, $|S_1^1\circ\beta\cdot\tilde{\varphi}|< C|\tilde{\varphi}|^3$, and therefore this term may be estimated in $L^4(S^1\times S^1)$ just like in Bourgain's estimates for a cubic nonlinearity. More precisely, by Lemma 3.6 we have

$$\left\| \psi \int_{0}^{t} S(t-\tau) \left(|\tilde{\varphi}|^{3}(\tau,\tilde{x}) \right) d\tau \right\|_{L^{4}(S^{1}\times S^{1})} \leq C(\delta B + B^{-1/4}) \|\psi \tilde{\varphi}\|_{L^{4}(S^{1}\times S^{1})}^{3}, \quad (51)$$

with B>0 as in the Lemma, δ is as before the time cut-off parameter, and C>0 is a uniform constant.

Fourth, using Lemma 3.4 and energy conservation we have

$$\left\| \psi \int_{0}^{t} S(t-\tau) \left(\tilde{\varphi} \int_{S^{1}} |\tilde{\varphi}|^{3} d\tilde{x}(\tau,\tilde{x}) \right) d\tau \right\|_{L^{4}(S^{1} \times S^{1})}$$

$$\leq C \int_{0}^{2\delta} \left\| \left| \tilde{\varphi}(\tau,\cdot) \int_{S^{1}} |\tilde{\varphi}(\tau,\cdot)|^{3} d\tilde{x} \right|_{L^{2}(S^{1})} d\tau$$

$$\leq C' \int_{0}^{2\delta} \int_{S^{1}} |\tilde{\varphi}(\tau,\cdot)|^{3} d\tilde{x} d\tau \leq C' \sqrt[4]{2} \delta^{1/4} ||\varphi||_{L^{4}(S^{1} \times S^{1})}^{3}.$$

$$(52)$$

Combining Equations (49)–(52) we have

$$||\psi\tilde{\varphi}||_{L^4(S^1\times S^1)} \le C((1+\delta)||\varphi||_{L^2(\mathbb{R}_x)} + (\delta B + B^{-1/4} + \delta^{1/4})||\psi\tilde{\varphi}||_{L^4(S^1\times S^1)}^3). \tag{53}$$

By choosing B large enough and then choosing δ small enough, in such a manner that δB is also small enough (all these choices depend only on the initial energy and the background geometry) we therefore may argue as Bourgain (a standard Picard iteration argument, see [B], p. 139) to obtain a uniform estimate on $||\tilde{\varphi}||_{L^4(\mathbb{R}_{t,loc}\times S^1)} = ||\varphi||_{L^4(\mathbb{R}_{t,loc}\times S^1)} = ||a||_{L^4([-\delta,\delta]\times S^1)}$. Global well-posedness (existence and uniqueness) in $L^4(\mathbb{R}_{t,loc}\times S^1)$ now follows from energy conservation.

To obtain global well-posedness for our original flow equation in $W^{k,2}$ we take k-1 derivatives of Equation (46). Unlike in the case of $M=\mathbb{R}$ (Section 2), for some of the terms we might need to use some interpolation inequalities already for the case k=2. In addition, compared to Bourgain's situation of the cubic NLS on S^1 we obtain in fact certain terms that are worse than those obtained by differentiating the cubic NLS. For example, for k=2 we obtain terms of order $|\tilde{\varphi}|^4$ and $|\tilde{\varphi}_{,x}\tilde{\varphi}^3|$. Such terms may be handled nevertheless. We carry this in some detail in the case k=2, omitting the details in the case $k\geq 3$ as they are similar.

Set $c := \tilde{\varphi}_{,\tilde{x}}$. Taking a derivative of Equation (46) we obtain

$$\sqrt{-1}c_{,t} = c_{,\tilde{x}\tilde{x}} + F$$

where

$$|F| < C(|\tilde{\varphi}|^4 + |c|(1+|\tilde{\varphi}|^2 + |\tilde{\varphi}|^3 + ||\tilde{\varphi}||_{L^1(S^1)}^3)).$$
(54)

As before we would like to obtain an $L^4(\mathbb{R}_{t,loc} \times S^1)$ estimate, this time for c. If we attempt to use Lemma 3.4 in order to handle the term $|\varphi|^4$ we would need to estimate

$$\int_0^t ||\varphi(\tau,\,\cdot\,)^4||_{L^2(S^1)} d\tau.$$

Using the Gagliardo-Nirenberg inequality [A], p. 93, we would have

$$||\varphi||_{L^{8}(S^{1})}^{4} \le c||\nabla_{x}\varphi||_{L^{4}(S^{1})}^{1/2}||\varphi||_{L^{4}(S^{1})}^{7/2}$$

and this can be estimated by $\epsilon(||\nabla_x \varphi||_{L^4(S^1)} + 1)^4 + C_{\epsilon}(||\varphi||_{L^4(S^1)} + 1)^4$, for appropriate $\epsilon, C_{\epsilon} > 0$. This is not enough for our purposes since we need to control the

coefficient multiplying the second term. Instead, we apply Lemma 3.6 which gives an improved estimate. Namely, we obtain

$$\left\| \psi \int_0^t S(t-\tau) \left(|\tilde{\varphi}|^4(\tau, \tilde{x}) \right) d\tau \right\|_{L^4(S^1 \times S^1)} \le C(B^{-1/4} + \delta B) \|\varphi\|_{L^{16/3}(\mathbb{R}_{t, loc} \times S^1)}^4. \tag{55}$$

By the Gagliardo-Nirenberg inequality, as before, this is therefore bounded by

$$C(B^{-1/4} + \delta B) \left(\epsilon ||\nabla_x \varphi||_{L^4(R_{t,loc} \times S^1)}^4 + C_{\epsilon} ||\varphi||_{L^4(R_{t,loc} \times S^1)}^4. \right)$$
 (56)

Next, to handle the term $|c||\tilde{\varphi}|^3 \leq \epsilon |c|^2 + C_{\epsilon}|\varphi|^6$, again by Lemma 3.6, we compute

$$\left| \left| \psi \int_0^t S(t-\tau) \left(|c\tilde{\varphi}|^3(\tau,\tilde{x}) \right) d\tau \right| \right|_{L^4(S^1 \times S^1)}$$

$$\leq C(B^{-1/4} + \delta B) \Big(C_{\epsilon'} ||\varphi||_{L^{8}(\mathbb{R}_{t,loc} \times S^{1})}^{6} + \epsilon' ||c||_{L^{8/3}(\mathbb{R}_{t,loc} \times S^{1})}^{2} \Big)
\leq C(B^{-1/4} + \delta B) \Big(C_{\epsilon'} C_{\epsilon''} ||\varphi||_{L^{4}(\mathbb{R}_{t,loc} \times S^{1})}^{6} + C_{\epsilon'} \epsilon'' ||c||_{L^{4}(\mathbb{R}_{t,loc} \times S^{1})}^{6} \Big)$$
(57)

$$+\epsilon' C_{\epsilon'''} ||\varphi||^2_{L^4(\mathbb{R}_{t,loc}\times S^1)} + \epsilon' \epsilon''' ||c||^2_{L^4(\mathbb{R}_{t,loc}\times S^1)}$$
.

The other terms in (54) are of lower order than the two terms we just estimated. Since we are free to choose appropriate B and δ , it follows from (55)–(57) and the Duhamel formula that the same Picard iteration type argument that was used earlier applies to our situation. Note that this estimate depends on the $L^2(S^1)$ norm of c, which we control under our assumption $\tilde{\varphi}(0) \in W^{1,2}(S^1)$ and on the $L^4(\mathbb{R}_{t,loc} \times S^1)$ norm of φ that we already control uniformly. This concludes the proof of Theorem 3.2.

4. Maps of the circle into a Kähler manifold

In this section we explain the difficulties that one encounters if one tries to apply the same methods to treat the case of maps from the circle to Kähler manifolds of arbitrary dimension $n \ge 1$.

For general n, one gets an expression for a solution of (13) given by the chronological exponential

$$e(t, x+1) = A^{-1}(t, x)e(t, x)$$

$$:= \lim_{n \to \infty} e^{-\frac{1}{n}B_{U}(t, x+1)} e^{-\frac{1}{n}B_{U}(t, x+\frac{n-1}{n})} \cdots e^{-\frac{1}{n}B_{U}(t, x+\frac{1}{n})} e(t, 0)$$
(58)

(see for example [DFN]).

First we observe that A(t,x) does not depend on x: indeed applying ∇_x to Equation (58) and using the fact that $\nabla_x e = 0$ we obtain $A_{,x} = 0$. From now on we simply write A(t).

Now $\Phi(t, x + 1) = A(t)\Phi(t, x)$. Since A(t) is unitary (and hence normal) it is unitarily diagonalizable and we set

$$A(t) = U(t)^* D(t) U(t),$$

with
$$U(t) \in U(n)$$
 and $D(t) = \operatorname{diag}(e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_n})$. Let
$$\tilde{\mathbf{\Phi}}(t,x) := U(t)^*D(t)^{-x}U(t)\mathbf{\Phi}(t,x) = A(t)^{-x}\mathbf{\Phi}(t,x).$$

The vector-valued function $\tilde{\Phi}$ is periodic in x. Moreover, by a computation similar to (44), so are all of its x-derivatives. We have

$$\Phi_{,t} = (A(t)^{x})_{,t}\tilde{\Phi} + A(t)^{x}\tilde{\Phi}_{,t},$$

$$\Phi_{,x} = U(t)^{*}D(t)^{x}\operatorname{diag}(\sqrt{-1}\theta_{i})U(t)\tilde{\Phi} + U(t)^{*}D(t)^{x}U(t)\tilde{\Phi}_{,x},$$

$$\Phi_{,xx} = U(t)^{*}D(t)^{x}\operatorname{diag}(-\theta_{i}^{2})U(t)\tilde{\Phi} + 2U(t)^{*}D(t)^{x}\operatorname{diag}(\sqrt{-1}\theta_{i})U(t)\tilde{\Phi}_{,x}$$

$$+U(t)^{*}D(t)^{x}U(t)\tilde{\Phi}_{,xx}.$$
(59)

It follows that

$$\sqrt{-1}\mathbf{\Phi}_{,t} - \mathbf{\Phi}_{,xx} = A(t)^x \left[\sqrt{-1}\tilde{\mathbf{\Phi}}_{,t} - \tilde{\mathbf{\Phi}}_{,xx} + (A(t)^x)_{,t}\tilde{\mathbf{\Phi}} \right]
-U(t)^* \operatorname{diag}(-\theta_i^2) U(t)\tilde{\mathbf{\Phi}} - 2U(t)^* \operatorname{diag}(\sqrt{-1}\theta_i) U(t)\tilde{\mathbf{\Phi}}_{,x} \right]$$
(60)

Therefore equations (40)-(41) may be rewritten as

$$\sqrt{-1}\tilde{\mathbf{\Phi}}_{,t} = \tilde{\mathbf{\Phi}}_{,xx} - (A(t)^x)_{,t}\tilde{\mathbf{\Phi}} + U(t)^* \operatorname{diag}(-\theta_i^2)U(t)\tilde{\mathbf{\Phi}}
+2U(t)^* \operatorname{diag}(\sqrt{-1}\theta_i)U(t)\tilde{\mathbf{\Phi}}_{,x} - A(t)^{-x} (\mathbf{Q} \cdot \mathbf{\Phi} + \mathbf{S} \cdot \mathbf{\Phi} - \mathbf{W} \cdot \mathbf{\Phi} + \mathbf{T} \cdot \mathbf{\Phi}).$$
(61)

Note that the last term is expressed in terms of Φ instead of Φ . However as far as the estimates are concerned this is not important since it involves no derivatives and the two vectors differ by a unitary transformation. Two problems now arise. First, one needs to obtain an estimate on the variation of the holonomy matrix A(t) along the flow. Such an estimate was available in the one-dimensional setting due to the Gauss-Bonnet theorem. Second, the matrix multiplying the first derivative term is not diagonal and so it is not clear how to eliminate this term.

Although this requires some work, and we will not attempt to provide the details here, the first difficulty may be overcome using the theory developed by Chacon and Fomenko for a non-commutative version of the Stokes' Theorem product integrals [CF] (see also the classical references [N, S]). To approach the second difficulty one may consider $\hat{\Phi} := U(t)^*D(t)^{-x}\Phi$ instead of $\tilde{\Phi}$. Then the matrix multiplying the first derivative of $\hat{\Phi}$ is diagonal. Therefore, we may apply the space-time transformation as in the Riemann surface case, however for each equation in the system separately. However, this introduces a new obstacle. Indeed, then one needs to control the time derivative of D(t) as well as of U(t). The main difficulty comes from the latter. In general, the unitary diagonalizing matrix does not vary smoothly (or even continuously) even when a family of matrices does (see [K], p. 111). Instead one may try to diagonalize A(t) smoothly. However, to the best of our knowledge, even given such a diagonalization, the problem is that even though the diagonalizing

matrix is then smooth one has essentially no control over its derivatives (i.e., estimates on these derivatives in terms of derivatives of A(t)). We hope to come back to this problem in the future. In some sense the two transformations (to $\tilde{\Phi}$ and to $\hat{\Phi}$) are dual to each other, and one may ask whether for higher-dimensional domains the two troublesome terms, namely the first derivative term and the derivative of the holonomy, may be a source for finite-time blow-up.

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